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Action Principle and Algebraic Approach to Gauge Transformations in Gauge Theories

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The action principle is used to derive, by an entirely *algebraic* approach, gauge transformations of the full vacuum-to-vacuum transition amplitude (generating functional) from the Coulomb gauge to arbitrary covariant gauges and in turn to the celebrated Fock–Schwinger (FS) gauge for the Abelian (QED) gauge theory without recourse to path integrals or to commutation rules and without making use of delta functionals. The interest in the FS gauge, in particular, is that it leads to Faddeev–Popov ghosts-free non-Abelian gauge theories. This method is expected to be applicable to non-Abelian gauge theories including supersymmetric ones.

KEY WORDS: action principle; gauge transformations; Coulomb gauge; Fock–Schwinger gauge.
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1. INTRODUCTION

About two decades ago, we have seen (Manoukian, 1986, 1987) that the very elegant action principle (Schwinger, 1951a,b, 1953a,b, 1954) may be used to quantize gauge theories in constructing the vacuum-to-vacuum transition amplitude and the Faddeev–Popov factor (Faddeev and Popov, 1967), encountered in non-Abelian gauge theories, was obtained *directly* from the action principle without much effort. No appeal was made to path integrals, no commutation rules were used, and there was not even the need to go into the well-known complicated structure of the Hamiltonian (Fradkin and Tyutin, 1970) in non-Abelian gauge theories. Of course path integrals are extremely useful in many respects and may be formally derived from the action principle cf. (Symanzik, 1954; Lam, 1965; Manoukian, 1985). We have worked in the Coulomb gauge, where the physical components are clear at the outset, to derive the expression for the vacuum-to-vacuum transition amplitude (generating functional) including the Faddeev–Popov

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factor in non-Abelian gauge theories. It is interesting to note also that the Coulomb gauge naturally arises (Faddeev and Jackiw, 1988; Ogawa *et al.*, 1986), see also (Jogleker and Mandal, 2002), in gauge field theories as constrained dynamics cf. (Henneaux and Teitelboim, 1992; Garcia and Vergara, 1996; Su, 2001). To make transitions of the generating functional to arbitrary covariant gauges, we have made use (Manoukian, 1986, 1987), in the process, of so-called δ functionals (Schwinger, 1972, 1973). The δ functionals, however, are defined as infinite dimensional continual integrals corresponding to the different points of spacetime and hence the gauge transformations were carried out in the spirit of path integrals.

The purpose of the present investigation is, in particular, to remedy the above situation involved with delta functionals, and we here derive the gauge transformations, providing explicit expressions, for the full vacuum-to-vacuum transition amplitude to the generating functionals of arbitrary covariant gauges and, in turn, to the celebrated Fock–Schwinger (FS) gauge $x^\mu A_\mu = 0$ (Fock, 1937), as well as the axial gauge $n^\mu A_\mu = 0$ for a fixed vector n^μ, for the Abelian (QED) gauge theory by an entirely *algebraic* approach dealing only with commuting (or anti-commuting) external sources. The interest in the FS gauge, in gauge theories, in general, is that it leads to Faddeev–Popov ghost-free theories, cf. (Kummer and Weiser, 1986), the gauge field may be expressed quite simply in terms of the field strength (Kummer and Weiser, 1986; Durand and Mendel, 1982) and it turns out to be useful in non-perturbative studies, cf. (Shifman *et al.*, 1979). Needless to say, the complete expressions of such generating functionals allow one to obtain gauge transformations of *all* the Green functions in a theory simply by functional differentiations with respect to the external sources coupled to the quantum fields in question and avoids the rather tedious treatment, but provides information on, the gauge transformation of diagram by diagram (Handy, 1979; Feng and Lam, 1996) occurring in a theory. A key point, whose importance cannot be overemphasized, in our analysis (Manoukian, 1986, 1987) is that, a priori, *no* restrictions are set on the external source(s) J^μ coupled to the gauge field(s), such as a $\partial_\mu J^\mu = 0$—restriction, so that *variations of the components of* J^μ *may be carried out independently*, until the entire analysis is completed. The present method is expected to be applicable to non-Abelian gauge theories including supersymmetric ones and the latter will be attempted in a forthcoming report. Some classic references which have set the stage of the investigation of the gauge problem in field theory are given in Landau and Khalatnikov (1954), Landau and Khalatnikov (1956), Johnson and Zumino (1959), Zumino (1960), Bialynicki-Birula (1968), Mills (1971), Slavnov (1972), Taylor (1971), Abers and Lee (1973), Wess and Zumino (1974), Salam and Strathdee (1974), Becchi *et al.* (1975), Utiyama and Sakamoto (1977). For more recent studies which are, however, more involved with field operator techniques and their gauge transformations may be found in Sardanashvily (1984), Kobe, (1985), Oh and Soo (1987), Sugano and Kimura (1990), Gastmans *et al.* (1996), Pons *et al.* (1997), Gastmans and Wu (1998), and Banerjee (2000).

2. GAUGE TRANSFORMATIONS

The Lagrangian density under consideration is given by a well-known expression (Manoukian, 1986, 1987)

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\left(\frac{\partial_\mu\overline{\psi}}{\mathrm{i}}\right)\gamma^\mu\psi - \overline{\psi}\gamma^\mu\frac{\partial_\mu\psi}{\mathrm{i}}\right] - m_0\overline{\psi}\psi$$

$$+ e_0\overline{\psi}\gamma_\mu\psi A^\mu + \overline{\eta}\psi + \overline{\psi}\eta + A_\mu J^\mu \qquad (1)$$

where $\overline{\eta}$, η, J^μ are external sources, and no restriction is set on J^μ (such as $\partial_\mu J^\mu = 0$) in order to carry out functional differentiations with respect to all of its components *independently*.

Our starting point is the vacuum-to-vacuum transition amplitude in the Coulomb gauge given by Manoukian (1986, 1987)

$$\langle 0_+|0_-\rangle = \exp\left[\mathrm{i}\int \mathscr{L}'_{\mathrm{I}}\right]\langle 0_+|0_-\rangle_0 \equiv F_{\mathrm{C}}[\eta, \bar{\eta}, J] \qquad (2)$$

$$\int \mathscr{L}'_{\mathrm{I}}(\eta, \bar{\eta}, J) = \int (\mathrm{d}x)\left(e_0\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma^\mu\frac{\delta}{\mathrm{i}\delta\bar{\eta}(x)}\frac{\delta}{\mathrm{i}\delta J^\mu x}\right) \qquad (3)$$

where

$$\langle 0_+\,|\,0_-\rangle_0 = \exp\left[\mathrm{i}\int(\mathrm{d}x)\,(\mathrm{d}x')\,\overline{\eta}(x)S_+(x-x')\eta(x')\right]$$

$$\times\exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)\,(\mathrm{d}x')\;J^\mu(x)D^{\mathrm{C}}_{\mu\nu}(x,x')J^\nu(x')\right] \qquad (4)$$

with $S_+(x-x')$ denoting the free electron propagator, and, in the momentum description, $(k, m = 1, 2, 3)$,

$$D^{\mathrm{C}}_{km}(q) = \left(\delta_{km} - \frac{q_k q_m}{\vec{q}^{\,2}}\right)\frac{1}{q^2 - \mathrm{i}\varepsilon} \qquad (5)$$

$$D^{\mathrm{C}}_{0k}(q) = 0 = D^{\mathrm{C}}_{k0}(q) \qquad (6)$$

$$D^{\mathrm{C}}_{00}(q) = -\frac{1}{\vec{q}^{\,2}}. \qquad (7)$$

We introduce the generating functional

$$F\big[\rho, \overline{\rho}, K; G\big] = \exp\left[\mathrm{i}\int\mathscr{L}'_{\mathrm{I}}(\rho, \overline{\rho}, K)\right]\exp\left[\mathrm{i}\int(\mathrm{d}x)\,(\mathrm{d}x')\,\overline{\rho}(x)S_+(x-x')\rho(x')\right]$$

$$\times\exp\left[\frac{\mathrm{i}}{2}\int(\mathrm{d}x)\,(\mathrm{d}x')\;K_\mu(x)D^{\mu\nu}_G(x,x')K_\nu(x')\right] \qquad (8)$$

where in the momentum description

$$D_G^{\mu\nu}(q) = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{1}{q^2 - \mathrm{i}\varepsilon} + q^\mu q^\nu G(q^2) \tag{9}$$

and $G(q^2)$ is arbitrary.

We show that

$$F_{\mathrm{C}}[\eta, \overline{\eta}, J] = \mathrm{e}^{\mathrm{i}W'} F\big[\rho, \overline{\rho}, K; G\big]\big|_{\rho=0,\overline{\rho}=0,K=0} \tag{10}$$

where

$$\begin{aligned} W' = &\int(\mathrm{d}x)\,\overline{\eta}(x)\exp\left[-\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu(x)}\right]\frac{\delta}{\mathrm{i}\delta\overline{\rho}(x)} \\ &+ \int(\mathrm{d}x)\,\frac{\delta}{\mathrm{i}\delta\rho(x)}\exp\left[\mathrm{i}e_0 a^\mu \frac{\delta}{\mathrm{i}\delta K^\mu(x)}\right]\eta\,(x) \\ &+ \int(\mathrm{d}x)\,((g^{\mu\sigma} - a^\mu\partial^\sigma)J_\sigma(x))\frac{\delta}{\mathrm{i}\delta K^\mu(x)} \end{aligned} \tag{11}$$

and

$$a^\mu = \left(0, \frac{\vec{\nabla}}{\nabla^2}\right) = g^{\mu k}\frac{\partial^k}{\nabla^2} \tag{12}$$

relating the Coulomb gauge to arbitrary covariant gauges.

To establish (10), we start from its right-hand side. We note, in a matrix notation, that

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}W'}\exp\left[\mathrm{i}\overline{\rho}S_+\rho\right]\exp\left[\frac{\mathrm{i}}{2}K_\mu D_G^{\mu\nu}K_\nu\right] \\ &\quad = \exp\left[\mathrm{i}\left(\overline{\rho} + \overline{\eta}\exp\left[-\mathrm{i}e_0 a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\right)S_+\left(\rho + \exp\left[\mathrm{i}e_0 a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\eta\right)\right] \\ &\quad \times\exp\left[\frac{\mathrm{i}}{2}\Big(K_\mu + (g_{\mu\sigma} - a_\mu\partial_\sigma)\,J^\sigma\Big)D_G^{\mu\nu}\Big(K_\nu + (g_{\nu\lambda} - a_\nu\partial_\lambda)J^\lambda\Big)\right] \end{aligned} \tag{13}$$

and since $\mathscr{L}_{\mathrm{I}}'(\rho, \overline{\rho}, K)$, is classical, is invariant under transformations $\rho(x) \to \rho(x)\exp\big(\mathrm{i}\Lambda(x)\big)$, $\overline{\rho}(x) \to \exp\big(-\mathrm{i}\Lambda(x)\big)\,\overline{\rho}\,(x)$ for an arbitrary numerical function $\Lambda(x)$, and we eventually set $\rho = 0$, $\overline{\rho} = 0$, the right-hand side of (10) becomes

$$\begin{aligned} &\exp\left[\mathrm{i}\int\mathscr{L}_{\mathrm{I}}'(\eta, \overline{\eta}, J)\right]\exp\left[\mathrm{i}\left(\overline{\eta}\exp\left[-\mathrm{i}e_0 a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\right)S_+\left(\exp\left[\mathrm{i}e_0 a^\mu\frac{\delta}{\mathrm{i}\delta K^\mu}\right]\eta\right)\right] \\ &\quad\times\exp\left[\frac{\mathrm{i}}{2}(K_\mu + (g_{\mu\sigma} - a_\mu\partial_\sigma)J^\sigma)D_G^{\mu\nu}(K_\nu + (g_{\nu\lambda} - a_\nu\partial_\lambda)J^\lambda)\right] \end{aligned} \tag{14}$$

with $K_\mu \to 0$. Now we use the identity

$$\exp\left[\mathrm{i}e_0\int(\mathrm{d}x)\left(\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma^\mu\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\partial_\mu\Lambda(x)\right)\right]\exp[\mathrm{i}\overline{\eta}S_+\eta]$$
$$= \exp\left[\mathrm{i}(\overline{\eta}\,\mathrm{e}^{\mathrm{i}e_0\Lambda})S_+(\mathrm{e}^{-\mathrm{i}e_0\Lambda}\eta)\right] \tag{15}$$

to rewrite the above expression as

$$\exp\left[\mathrm{i}e_0\int(\mathrm{d}x)\left(\frac{\delta}{\mathrm{i}\delta\eta(x)}\gamma_\mu\frac{\delta}{\mathrm{i}\delta\overline{\eta}(x)}\left(g^{\mu\sigma}-a^\sigma\partial^\mu\right)\frac{\delta}{\mathrm{i}\delta K^\sigma(x)}\right)\right]\exp[\mathrm{i}\overline{\eta}S_+\eta]$$
$$\times\exp\left[\frac{\mathrm{i}}{2}\left(K_\mu+\left(g_{\mu\sigma}-a_\mu\partial_\sigma\right)J^\sigma\right)D_G^{\mu\nu}\left(K_\nu+\left(g_{\nu\lambda}-a_\nu\partial_\lambda\right)J^\lambda\right)\right] \tag{16}$$

which for $K_\mu \to 0$ reduces to the left-hand side of (10) *since*

$$\left(g_{\mu\sigma}-a_\mu\partial_\sigma\right)D_G^{\mu\nu}\left(g_{\nu\lambda}-a_\nu\partial_\lambda\right)=D^{\mathrm{C}}_{\sigma\lambda}. \tag{17}$$

Almost an identical analysis as above shows, by noting in the process,

$$\left(g_{\mu\sigma}-\widetilde{a}_\mu\partial_\sigma\right)D_G^{\mu\nu}\left(g_{\nu\lambda}-\widetilde{a}_\nu\partial_\lambda\right)=\left(D_0\right)_{\sigma\lambda}\equiv D^{\mathrm{L}}_{\sigma\lambda} \tag{18}$$

with

$$\widetilde{a}_\mu=\frac{\partial_\mu}{\Box},\ \Box\equiv\partial_\mu\partial^\mu \tag{19}$$

where the right-hand side of (18) defines the photon propagator in the Landau gauge, with G in (9) set equal to zero, that

$$F[\eta,\overline{\eta},J;G=0]=\mathrm{e}^{\mathrm{i}\widetilde{W}'}F[\rho,\overline{\rho},K;G]\big|_{\rho=0,\overline{\rho}=0,K=0} \tag{20}$$

where $\widetilde{W}'$ is given by the expression defined in (11) with a^μ in it simply replaced by $\widetilde{a}^\mu$, thus relating the Landau gauge to arbitrary covariant gauges.

The Fock–Schwinger gauge $x^\mu A_\mu = 0$, allows one to write

$$A^0=\frac{x^kA_k}{x^0} \tag{21}$$

which upon substitution in (1), and varying $\mathscr{L}$ with respect to A^k yields

$$\partial_\mu F^{\mu k}-\frac{x^k}{x^0}\partial_\mu F^{\mu 0}=-j^k+j^0\frac{x^k}{x^0} \tag{22}$$

where

$$j^\mu=e_0\overline{\psi}\gamma^\mu\psi+J^\mu. \tag{23}$$

We note that (22) holds true with k replaced by 0 in it giving $0=0$, i.e., we may rewrite (22) as

$$\partial_\mu F^{\mu\nu}-\frac{x^\nu}{x^0}\partial_\mu F^{\mu 0}=-j^\nu+j^0\frac{x^\nu}{x^0}\equiv S^\nu. \tag{24}$$

By taking the derivative ∂_ν of (24), we may solve for $(\partial_\mu F^{\mu 0})/x^0$,

$$-\frac{\partial_\mu F^{\mu 0}}{x^0} = (\partial\, x)^{-1}\partial_\sigma\left(-j^\sigma + j^0\frac{x^\sigma}{x^0}\right) \tag{25}$$

which upon substituting in (24) gives

$$\partial_\mu F^{\mu\nu} = -\left[g^{\nu\sigma} - x^\nu(\partial\, x)^{-1}\partial^\sigma\right]j_\sigma. \tag{26}$$

By taking $\nu = k$, and taking the derivative ∂_k of (26), we may write

$$-\partial_0 A^0 = \frac{1}{\nabla^2}\left(\partial_0^2\,\partial_k A^k + \partial_k S^k\right) \tag{27}$$

which when substituted in (26) gives

$$A^\nu = \Box^{-1}S^\nu + \frac{\partial^\nu}{\nabla^2}\left(\partial_k A^k - \frac{1}{\Box}\partial_k S^k\right). \tag{28}$$

That is, A^ν is of the form

$$A^\nu = \Box^{-1}S^\nu + \partial^\nu a. \tag{29}$$

For $\nu = k$, and multiplying (29) by x^k/x^0, we have from (21)

$$A^0 = \frac{x^k}{x^0}\Box^{-1}S^k + \frac{x^k}{x^0}\partial^k a. \tag{30}$$

On the other hand, directly from (29) with $\nu = 0$ in it,

$$A^0 = \Box^{-1}S^0 + \partial^0 a \tag{31}$$

which upon comparison with (30) leads to

$$x\partial a = -x^\mu\Box^{-1}S_\mu. \tag{32}$$

From (29), (32) and the definition of S^ν in (24), we obtain

$$A^\nu = -\frac{1}{\Box}\left(g^{\nu\mu} - \partial^\nu\frac{1}{x\,\partial + 2}x^\mu\right)\left(g_{\mu\sigma} - x_\mu\frac{1}{\partial x}\partial_\sigma\right)j^\sigma \tag{33}$$

where we have noted that $\partial x = 4 + x\partial$. It is straightforward to check from (33) that $x_\nu A^\nu = 0$ is indeed satisfied.

To establish the transformation from covariant gauges to the FS gauge, we have to pull $\Box^{-1}$ in (33) between the two round brackets. To this end we note that

$$\Box x\partial = (x\partial + 2)\,\Box \tag{34}$$

and hence

$$(\Box\, x\partial)^{-1} = (x\partial)^{-1}\Box^{-1} = \Box^{-1}(x\partial + 2)^{-1} \tag{35}$$

i.e.,

$$\frac{1}{\Box}\frac{1}{x\partial+2}=\frac{1}{x\partial}\frac{1}{\Box}. \tag{36}$$

We may also use the identity

$$\frac{1}{\Box}x^{\mu}=x^{\mu}\frac{1}{\Box}-2\frac{\partial^{\mu}}{\Box} \tag{37}$$

and since ∂^{μ} when applied to the second factor in (33) gives

$$\partial^{\mu}\left(g_{\mu\sigma}-x_{\mu}\frac{1}{\partial x}\partial_{\sigma}\right)=0. \tag{38}$$

We obtain from (36)–(38), (33)

$$A^{\nu}=\left(g^{\nu\mu}-\partial^{\nu}\frac{1}{x\partial}x^{\mu}\right)\frac{1}{(-\Box)}\left(g_{\mu\sigma}-x_{\mu}\frac{1}{\partial x}\partial_{\sigma}\right)j^{\sigma}. \tag{39}$$

Now we invoke the transversality property in (38) to *rewrite* (39) as

$$A^{\nu}=\left(g^{\nu\mu}-\partial^{\nu}\frac{1}{x\partial}x^{\mu}\right)\frac{1}{(-\Box)}\Big[g_{\mu\rho}-H(\Box)\partial_{\mu}\partial_{\rho}\Big]\left(g^{\rho\sigma}-x^{\rho}\frac{1}{\partial x}\partial^{\sigma}\right)j_{\sigma} \tag{40}$$

where $H(\Box)$ is *arbitrary* on account of (38).

It remains to set

$$g^{\rho\sigma}-x^{\rho}\frac{1}{\partial x}\partial^{\sigma}=O^{\rho\sigma} \tag{41}$$

and note that for the factor multiplying j_{σ} on the right-hand side of (40),

$$\langle x|(\bullet)|x'\rangle=\int(\mathrm{d}x'')\,(\mathrm{d}x''')\langle x''|O^{\mu\nu}|x\rangle\langle x''|(D_{\mathrm{H}})_{\mu\rho}|x'''\rangle\langle x'''|O^{\rho\sigma}|x'\rangle \tag{42}$$

where, as shown in the appendix, we have noted that

$$\langle x|\partial^{\nu}(x\partial)^{-1}x^{\mu}|x'\rangle=\langle x'|x^{\mu}(\partial x)^{-1}\partial^{\nu}|x\rangle \tag{43}$$

and we recognize $\langle x''|(D_H)_{\mu\rho}|x'''\rangle$ to have the very general structure in (9). Hence we may write, as in (10),

$$F_{\mathrm{FS}}[\eta,\overline{\eta},J]=\mathrm{e}^{\mathrm{i}W''}F[\rho,\overline{\rho},K;G]|_{\rho=0,\overline{\rho}=0,K=0} \tag{44}$$

where W'' is given by (11) with a^{μ} in the latter replaced by $x^{\mu}(\partial x)^{-1}$. [For interpretation of $x^{\mu}(\partial x)^{-1}\partial^{\nu}$ see the appendix and also Kummer and Weiser (1986).]

The axial gauge $n^{\mu}A_{\mu}=0$, with n^{ν} a fixed vector, is handled similarly, with A^{ν} in (39) now replaced by

$$A^{\nu}=\left(g^{\nu\mu}-\partial^{\nu}\frac{1}{n\,\partial}n^{\mu}\right)\frac{1}{(-\Box)}\left(g_{\mu\sigma}-n_{\mu}\frac{1}{n\,\partial}\partial_{\sigma}\right)j^{\sigma} \tag{45}$$

and a similar expression as in (44) holds with a^μ in (10) replaced by $n^\mu(n\,\partial)^{-1}$ in it.

3. CONCLUSION

We have seen that the algebraic method developed in this work solves the gauge transformation problem relating generating functionals in different gauges starting from the vacuum-to-vacuum transition amplitude in the Coulomb gauge. Needless to say, their transformation rules give the transformations of *all* the Green functions encountered in the theory and avoids unnecessary tedious steps otherwise involved. The simplicity and the power of the method is evident and it is expected to be applicable to non-Abelian gauge theories, with (Manoukian, 1986, 1987) or without Faddeev–Popov ghosts, as well as to supersymmetric theories. We have not, however, touched upon uniqueness problems such as the Gribov ambiguity (Gribov, 1978; Zwanziger, 1981). This and extensions to non-Abelian cases and supersymmetric theories will be attempted in a forthcoming report.

APPENDIX

For an explicit derivation of (43), we multiply ∂^ν by $-\mathrm{i}$ and write

$$\partial^\nu(x\partial)^{-1}x^\mu = (xp+1)^{-1}p^\nu x^\mu = \sum_{n=0}^{\infty}(-1)^n(xp)^n p^\nu x^\mu \tag{A.1}$$

upon moving, in the process, p^ν to the right. Using the identity

$$(x^\mu p_\mu)_{\mathrm{op}} = \int (\mathrm{d}x)\frac{(\mathrm{d}p)}{(2\pi)^4}\,|x\rangle\langle p|\,xp\;\mathrm{e}^{\mathrm{i}xp} \tag{A.2}$$

we note that

$$(xp)^n = \int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}x_i p_i\right]\mathrm{e}^{\mathrm{i}x_n(p_n-p_{n-1})}\mathrm{e}^{\mathrm{i}x_{n-1}(p_{n-1}-p_{n-2})}\ldots\mathrm{e}^{\mathrm{i}x_1p_1}\,|x_1\rangle\langle p_n| \tag{A.3}$$

and hence

$$\langle x|\partial^\nu(x\partial)^{-1}x^\mu|x'\rangle = \sum_{n=0}^{\infty}(-1)^n\int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}x_i p_i\right]p_n^\nu x'^\mu\delta(x-x_1)$$
$$\times\mathrm{e}^{\mathrm{i}x_n(p_n-p_{n-1})}\mathrm{e}^{\mathrm{i}x_{n-1}(p_{n-1}-p_{n-2})}\ldots\mathrm{e}^{\mathrm{i}x_1p_1}\mathrm{e}^{-\mathrm{i}p_nx}. \tag{A.4}$$

This may be rewritten in an equivalent form by making the change of variables

$$x_1 = y_n,\ldots,x_n = y_1;\quad p_1 = -q_n,\ldots,p_n = -q_1 \tag{A.5}$$

leading to

$$\langle x|\partial^{\nu}(x\partial)^{-1}x^{\mu}|x'\rangle = -\sum_{n=0}^{\infty}\int\left[\prod_{i=1}^{n}(\mathrm{d}y_i)\frac{(\mathrm{d}q_i)}{(2\pi)^4}y_i q_i\right]x'^{\mu}q_1^{\nu}\delta(y_n - x)$$
$$\times \mathrm{e}^{\mathrm{i}xq_1}\mathrm{e}^{\mathrm{i}y_1(q_2-q_1)}\mathrm{e}^{\mathrm{i}y_2(q_3-q_2)}\ldots\mathrm{e}^{-\mathrm{i}y_n q_n}. \tag{A.6}$$

On the other hand,

$$\begin{aligned}\langle x|x^{\mu}(\partial x)^{-1}\partial^{\nu}x'\rangle &= \langle x|x^{\mu}p^{\nu}(p\,x-1)^{-1}|x'\rangle \\ &= -\sum_{n=0}^{\infty}\langle x|x^{\mu}p^{\nu}(p\,x)^{n}|x'\rangle\end{aligned} \tag{A.7}$$

and

$$(p^{\mu}x_{\mu})_{\mathrm{op}} = \int(\mathrm{d}x)\frac{(\mathrm{d}p)}{(2\pi)^4}\,|p\rangle\langle x|\,p x\mathrm{e}^{-\mathrm{i}px} \tag{A.8}$$

$$(p\,x)^{n} = \int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}p_i x_i\right]\mathrm{e}^{\mathrm{i}x_1(p_2-p_1)}\ldots\mathrm{e}^{\mathrm{i}x_{n-1}(p_n-p_{n-1})}\mathrm{e}^{-\mathrm{i}x_n p_n}\,|p_1\rangle\langle x_n| \tag{A.9}$$

leading to

$$\langle x|x^{\mu}(\partial x)^{-1}\partial^{\nu}x'\rangle = -\sum_{n=0}^{\infty}\int\left[\prod_{i=1}^{n}(\mathrm{d}x_i)\frac{(\mathrm{d}p_i)}{(2\pi)^4}p_i x_i\right]x^{\mu}p_1^{\nu}\delta(x_n - x')$$
$$\times\,\mathrm{e}^{\mathrm{i}xp_1}\mathrm{e}^{\mathrm{i}x_1(p_2-p_1)}\ldots\mathrm{e}^{\mathrm{i}x_{n-1}(p_n-p_{n-1})}\mathrm{e}^{-\mathrm{i}x_n p_n} \tag{A.10}$$

which upon comparison with (A.6) establishes (43).

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